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RINGS IN WHICH EVERY NONZERO S-WEAKLY PRIME IDEAL IS S-PRIME

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Abstract. In this paper, we introduce and study a new class of rings with multiplicative subset S which we'll call S-WP-rings. A ring R with a multiplicative subset S is said to be S-WP-ring if every nonzero S-weakly prime ideal is S-prime. We next study the possible transfer of the properties of being S-WP-ring in the homomorphic image, in the localization, in the trivial ring extensions and the amalgamated algebra along an ideal introduced and studied by the authors of [5, 6, 7, 8]. Our results allow us to construct new original class of S-WP-rings subject to various ring theoretical properties.

1. Introduction

Our aim is to introduce and study the class of rings in which every nonzero S-weakly prime ideal is S-prime. Throughout this paper, all rings considered are assumed to be commutative with non-zero identity and all modules are nonzero unital. In [10], A. El Khalfi, N. Mahdou and Y. Zahir introduced the concept of WP-rings. A ring A is called WP-ring if every nonzero weakly prime ideal is prime. Recently, the concept of S-property has an important place in commutative algebra and they draw attention by several authors. The S-weakly prime ideals introduced by the authors of [1, 18] is a generalization of the work of A. Hamed and A. Malek in [12]. Following [18] a proper ideal P is said to be S-weakly prime (where $S \subseteq A$ multiplicative set, and $P \cap S = \varnothing$) if there exists $s \in S$ such that the following condition holds for every $a, b \in A$: $0 \neq ab \in P$ implies that either $sa \in P$ or $sb \in P$. We denote $\sqrt{0}$ is the set for all nilpotent elements of A; $\mathrm{Ann}(I)$ or $(0 : I)$ denote the annihilator of an ideal I; $Reg(A)$ denotes the set of all regular elements of A. If A is an integral domain, we denote its quotients field by $qf(A)$.

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Clearly, every S-prime ideal is S-weakly prime, but the converse is not true in general. For example, the zero ideal is a weakly prime (by definition) and so, it is S-weakly prime however is not S-prime if we assume that the ring considered is not an integral domain. There is no investigation on the following natural question: when every nonzero S-weakly prime ideal is S-prime?. For this, it is could be interest to study a class of rings satisfying the above question. We focus our attention to this study instead of class of rings in which every S-weakly prime ideal is S-prime. Let R be a ring and E an R-module. Then $R \propto E$, the trivial ring extension of R by E, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by $(a,e)(b,f) := (ab, af + be)$ for all $a, b \in R$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the idealization $R(+)E$) (see [14, 11, 4, 16]).

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following sub-ring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\},$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna et al. [6, 8]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana [7] and denoted by $A \bowtie I$).

This paper consists of three sections including introduction. In section 2, we investigate the transfer of S-WP-ring property to localization and homomorphic image. We next give the necessary conditions of different from nilpotent ideal to be a S-weakly prime which is not S-prime, this result allows us to give a characterization of S-WP-rings. In section 3, we study the possible transfer of the properties of being S-WP-ring in the amalgamated algebra along an ideal introduced and the trivial ring extensions.

2. **Basic results**

In this section, we introduce the S-WP-rings and we next give some properties of the notion. Our first definition in this section is given as follows.

Definition 2.1. *A ring A with multiplicative subset S is called S-WP-ring if every nonzero S-weakly prime ideal is S-prime.*

Remarks 2.2. *(1) If $S \subseteq U(A)$, A is an $S-W-P$-ring if and only if it is WP-ring.*

(2) If A is an integral domain, then A is an S-WP-ringfor every multplicative subset S of R.

It is clear that if $S_1, S_2, \cdots, S_n$ are multiplicative subsets of the rings $R_1, R_2, \cdots, R_n$ respectively, then $\prod_{i=1}^{n} S_i$ is a multiplicative subset of the ring $R = \prod_{i=1}^{n} R_i$. Next, we study the stability of the S-WP-ring under direct product.

Proposition 2.3. *Let $n \geq 2$ be an integer and $S_1, S_2, \cdots, S_n$ be multiplicative subsets of the rings $R_1, R_2, \cdots, R_n$ respectively. Set $R = \prod_{i=1}^{n} R_i$ and $S = \prod_{i=1}^{n} S_i$. Then R is always an S-WP-ring.*

Proof. It is enough to prove that this proposition is true for $n = 2$, the general case is established by induction on n. If P is a non-zero $(S_1 \times S_2)$-weakly prime ideal of the ring $R_1 \times R_2$, then P is $(S_1 \times S_2)$-prime by [18, Theorem 2.4]. Therefore, $R_1 \times R_2$ is an S-WP-ring. □

Next, let A be a ring, S be a multiplicative subset of A and I be an ideal of A. Assume that $I \cap S = \emptyset$. Notice that $\overline{S} = \{s + I \mid s \in S\}$ is a multiplicative subset of A/I. It is easy to verify that if $(P/I) \cap \overline{S} = \emptyset$, then $P \cap S = \emptyset$.

The following proposition establishes the transfer of the S-WP-ring property to homomorphic image.

Proposition 2.4. *Let A ba a ring with multiplicative subset S and I be an S-weakly prime ideal of A. If A is an S-WP-ring, then A/I is an $\overline{S}$-WP-ring.*

Proof. Assume that A is an S-WP-ring, of A and let J be a nonzero $\overline{S}$-weakly prime ideal of A/I. Then if $P/I \cap \overline{S} = \emptyset$, $J = P/I$ where P is a nonzero S-weakly prime ideal of A by [18, Prposition 2.6]. As A is an S-WP-ring, we get P is S-prime and so P/I is an $\overline{S}$-prime ideal of A/I by [12, Proposition 3]. Therefore, A/I is an $\overline{S}$-WP-ring. Now, if $(P/I) \cap \overline{S} \neq \emptyset$, then the desired result is obvious. □

Corollary 2.5. *Let A be a ring such that $S \subset U(R)$ and I is weakly prime. If A is a WP-ring, then A/I is a WP-ring.*

Proof. Obvious by Proposition 2.4. □

Proposition 2.6. *Let A be a ring and S be a multiplicative subset of R consist of regular elements. If $S^{-1}A$ is a WP-ring, then every nonzero S-weakly prime ideal disjoint of S is S-prime. The converse is true if* *the condition holds: for any ideal* $S^{-1}P$ *of* $S^{-1}A$*, there exists* $s \in S$ *such that* $S^{-1}P \cap A = (P : s)$*.*

Proof. Let P be a nonzero S-weakly prime ideal of A such that $I \cap S = \emptyset$. Then $S^{-1}P$ is weakly prime ideal of $S^{-1}A$ and $S^{-1}P \cap A = (P : s)$ for some $s \in S$ by [1, Proposition 2.14]. Since $S^{-1}A$ is WP-ring, we get $S^{-1}P$ is prime. Now, we claim that P is an S-prime ideal of A. Let $a, b \in A$ such that $ab \in P$. Since $\frac{a}{1}\frac{b}{1} \in S^{-1}P$, we get $\frac{a}{1} \in S^{-1}P$ or $\frac{b}{1} \in S^{-1}P$. But If $\frac{a}{1} \in S^{-1}P$, then $\frac{a}{1} = \frac{x}{s}$ for some $x \in P$ and for some $s \in S$, thus $tas = tx \in P$ for some $t \in S$. Now, we get $sa \in P$ for some $s \in S$. Therefore, P is an S-prime ideal of A.

Now assume that for any ideal $S^{-1}P$ of $S^{-1}A$, there exists $s \in S$ such that $S^{-1}P \cap A = (P : s)$. Then $S^{-1}P \cap A = (P : s)$. Let $S^{-1}P$ be a weakly prime ideal of $S^{-1}A$ and let $S^{-1}P \cap A = (P : s)$ for some $s \in S$. Following the [1, Proposition 2.14], we get P is an S-weakly prime ideal of A ($P \cap S = \emptyset$). By hypothesis, we get P is S-prime. Hence $S^{-1}P$ is a prime ideal of $S^{-1}A$ by [12, Remark 1], we are done. □

The next Propositions 2.7 and 2.8 establish when the Nilradical of a ring is S-prime.

Proposition 2.7. *If* P *is an* S*-weakly prime ideal of* A*, then either* $sP \subseteq \sqrt{0}$ *for some* $s \in S$ *or there exists* $s \in S$ *such that for every* $x \in \sqrt{0}$*, we get* $s^n x \in P$ *for some* n*.*

Proof. Assume that P is an S-weakly prime ideal of A. If P is not S-prime, then $s^2P^2 = 0$ for some $s \in S$ by [18, Theoreme 2.3]. Let $x \in sP$, so $x = sx'$ for some $x' \in P$. But we have $sx'sx' = (sx')^2 = 0$ and so $sx' \in \sqrt{0}$. Thus $sP \subseteq \sqrt{0}$ for some $s \in S$. Now, if P is S-prime, then there exists $s \in S$ such that for every $x \in \sqrt{0}$, we get $s^n x \in P$ for some n, as desired. □

Proposition 2.8. *Let* A *be a ring. If* $\sqrt{0}$ *is* S*-weakly prime such that* $Ann(\sqrt{0}) \subseteq \sqrt{0}$*, then* $\sqrt{0}$ *is* S*-prime.*

Proof. Let $a, b \in A$ such that $ab \in \sqrt{0}$. If $ab \neq 0$, then there exists $s \in S$ such that $sa \in \sqrt{0}$ or $sb \in \sqrt{0}$. Since $\sqrt{0}$ is an S-weakly prime ideal. Then we may assume that $ab = 0$. If $a\sqrt{0} \neq 0$, then there exists $r \in \sqrt{0}$ such that $ar \neq 0$ and so $0 \neq (b + r)a \in \sqrt{0}$. There exists also $s \in S$ such that $s(b + r) \in \sqrt{0}$ or $sb \in \sqrt{0}$. Now assume that $a\sqrt{0} = 0$. Likewise $b\sqrt{0} = 0$ and so $a \in Ann(\sqrt{0})$ implies that $a \in \sqrt{0}$. Hence $sa \in Ann(\sqrt{0})$ for some $s \in S$, as desired. □

Let R be a ring and $\mathfrak{q}$ be a proper ideal. Let S be a multiplicative subset of R. For an element $s \in S$, we set $s\text{-}tor(R/\mathfrak{q}) := \{r \in R \mid sr \in \mathfrak{q}\}$.

Recall from [18] that if R is a ring and S be a multiplicative subset of R. A proper ideal $\mathfrak{q}$ is said to be S-(weakly)-prime if there exists $s \in S$ such that for every $a, b \in R$, we have: $ab(\neq 0) \in \mathfrak{q}$ implies that $sa \in \mathfrak{q}$ or $sb \in \mathfrak{q}$. In this case, we denote by $(\mathfrak{q}, s)$ this ideal.

The following theorem characterizes the S-WP-rings.

Theorem 2.9. *Let R be a ring and S be a multiplicative subset. The following are equivalent:*

(1) R is an S-WP-rings,
(2) Every S-weakly prime ideal $(\mathfrak{q}, s)$, the following condition holds: For every $a \in Z(R)$, either $a \in s\text{-}tor(R/\mathfrak{q})$ or $(0 : a) \subset s\text{-}tor(R/\mathfrak{q})$,
(3) Every S-weakly prime ideal $(\mathfrak{q}, s)$, the following condition holds: For every $a \in Z(R)$, either $sa \in \mathfrak{q}$ or $s(0 : a) \subset \mathfrak{q}$.

The proof of Theorem 2.9 follows immediately from the following lemma.

Lemma 2.10. *Let R be a ring and S its multiplicative subset. The following are equivalent for a proper ideal $\mathfrak{q}$ of R:*

(1) $(\mathfrak{q}, s)$ is S-weakly-prime ideal such that the following condition holds: For every $a \in Z(R)$, either $a \in s\text{-}tor(R/\mathfrak{q})$ or $(0 : a) \subset s\text{-}tor(R/\mathfrak{q})$.
(2) $(\mathfrak{q}, s)$ is S-prime ideal.

Proof. The Lemma is obvious if we assume that R is a domain. It is established when $Z(R) \neq 0$.

(1) $\Rightarrow$ (2) Let $a, b \in R$ such that $ab = 0$. Our aim is to claim that either $sa \in \mathfrak{q}$ or $sb \in \mathfrak{q}$.

Case 1: If at least one of a or b is zero, then the result is trivial.

Case 2: Assume that both a and b are non-zero elements of R. Then $a \in Z(R) \setminus \{0\}$. By hypothesis, either $a \in s\text{-}tor(R/\mathfrak{q})$ or $(0 : a) \subset s\text{-}tor(R/\mathfrak{q})$, that is, $sa \in \mathfrak{q}$ or $sb \in \mathfrak{q}$ since $b \in (0 : a)$. Therefore, $\mathfrak{q}$ is S-prime ideal.

(2) $\Rightarrow$ (1) It is enough to check the following condition "for every $a \in Z(R) \setminus \{0\}$, either $a \in s\text{-}tor(R/\mathfrak{q})$ or $(0 : a) \subset s\text{-}tor(R/\mathfrak{q})$", since every S-prime ideal is S-weakly prime. Let $a \in Z(R) \setminus \{0\}$. So, if $a \in s\text{-}tor(R/\mathfrak{q})$, then as desired. If $a \notin s\text{-}tror(R/\mathfrak{q})$, then we claim that $(0 : a) \subset s\text{-}tor(R/\mathfrak{q})$. Let $b \in (0 : a)$, then $ab = 0$ and so either $sa \in \mathfrak{q}$ or $sb \in \mathfrak{q}$ since $\mathfrak{q}$ is assumed S-prime. But $a \notin s\text{-}tor(R/\mathfrak{q})$, then necessarily $sb \in \mathfrak{q}$, i.e., $b \in s\text{-}tor(R/\mathfrak{q})$. Therefore, $(0 : a) \subset s\text{-}tor(R/\mathfrak{q})$. This proof is completed.

□

Proof of Theorem 2.9
This follows immediately from Lemma 2.10 □.

Next, we will give a condition for which the S-weakly prime ideals and weakly-prime are the same. For this purpose, we recall the following Definition 2.11.

Definition 2.11. [17, Definition 1.6.10] *Let R be a ring and S be a multiplicative subset. An R-module M is said to be S-torsion free if every $s \in S$ and every $x \in M$ such that $sx = 0$, we get $x = 0$.*

The following Theorem 2.12 links the S-weakly prime ideals with weakly prime.

Theorem 2.12. *Let R be a ring and S be a multiplicative subset and $s \in S$. If $\mathfrak{q}$ is a proper ideal of R such that $R/\mathfrak{q}$ is an S-torsion-free R-module, the the following are equivalent:*

(1) $(\mathfrak{q}, s)$ is S-weakly-prime,
(2) $\mathfrak{q}$ is weakly-prime.

Proof. (2) $\Rightarrow$ (1) This is obvious.

(1) $\Rightarrow$ (2) Let $a, b \in R$ such that $ab \neq 0$ and $ab \in \mathfrak{q}$. Our aim is to show that either $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. Denoted by $\overline{x} := x + \mathfrak{q}$ in $R/\mathfrak{q}$ for every $x \in R$), since $\mathfrak{q}$ is S-weakly prime, then either $sa \in \mathfrak{q}$ or $sb \in \mathfrak{q}$, i.e., $s\overline{a} = \overline{0}$ or $s\overline{b} = \overline{0}$. But $R/\mathfrak{q}$ is an S-torsion-free R-module, implies that either $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, i.e., $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$, as desired $\mathfrak{q}$ is weakly-prime. □

Next, we study the possible transfer of the properties of being S-WP-ring in the trivial ring extensions. It is known from [3] that if S is a multiplicative subset of A, then $S \propto E$ is a multiplicative subset of $A \propto E$. We starts this part by characterization of the ideal of the form $I \propto F$ be S-weakly prime ideal.

Theorem 2.13. *Let A be a ring, I be an ideal of A, F a submodule of E and S a multiplicative subset of A such that $S \subseteq Reg(A)$ and $S \cap Ann(F) = \emptyset$. Then $I \propto F$ is $(S \propto 0)$-weakly prime if and only if the following condition holds:*

(1) I is S-weakly prime ideal of A and $s^2 I^2 = 0$ when $SE \nsubseteq F$ for every $s \in S$;
(2) If $b \in A$ and $s'f \notin F$, then $bf = 0$ or $bf \notin F$;
(3) If $ab \neq 0$ and $s'a \in I, s'b \notin I$, then $be \notin F$ for each $se \notin F$;
(4) If $ab = 0$, then
(i) $af + be = 0$ or $af + be \notin F$ for each $s'e, s'f \notin F$;
(ii) If $s'a \notin I$ and $s'b \notin I$, then $a \in Ann(F)$ and $b \in Ann(F)$.

Proof. Assume that $I \propto F$ is an $(S \propto 0)$-weakly prime ideal of $A \propto E$. Let $a, b \in A$ such that $0 \neq ab \in I$, then $(0,0) \neq (a,0)(b,0) \in I \propto F$ implies that there exists $(s,0) \in S \propto 0$ such that $(s,0)(a,0) \in I \propto F$ or $(s,0)(b,0) \in I \propto F$. So $sa \in I$ or $sb \in I$ and hence I is an S-weakly prime ideal of A. Assume that $SE \nsubseteq F$ for some $s \in S$, whence $((s,0)I \propto F)^2 = 0$ by [18, Theorem 2.3] and so $S^2I^2 = 0$. Consequently (1) holds. Now, let $b \in A$ and $s'f \notin F$ and assume that $bf \neq 0$ and $bf \in F$, then $(0,0) \neq (b,f)(0,f) \in I \propto F$ implies there exists $(s',0)(b,f) \in I \propto F$ or $(s',0)(0,f) \in I \propto F$. But neither $(s',0)(b,f) \in I \propto F$ nor $(s',0)(0,f) \in I \propto F$ a contradiction, so (2) holds. Let $s'a \in I$ and $s'b \notin I$ such that $ab \neq 0$ and assume that $be \in F$ for each $s'e \notin F$, then $(0,0) \neq (a,e)(b,0) = (ab,be) \in I \propto F$ and so $(s',0)(b,0) \in I \propto F$ or $(s',0)(a,e) \in I \propto F$, but neither $(s_0,0)(b,0) \in I \propto F$ nor $(s',0)(a,e) \in I \propto F$, a desired contradiction, so (3) holds. Assume that $ab = 0$, we pick $s'e, s'f \notin F$ such that $af + be \neq 0$ and $af + be \in F$, then $(0,0) \neq (a,e)(b,f) = (ab, af + be) \in I \propto F$ but neither $(s',0)(a,e) \in I \propto F$ nor $(s',0)(b,f) \in I \propto F$ a contradiction. Finally, let $s'a \notin I$ and $s'b \notin I$. Assume that there exists $e \in F$ such that $ae \neq 0$, then $(0,0) \neq (a,0)(b,e) = (ab, ae) \in I \propto F$ and but neither $(s',0)(a,0) \in I \propto F$ nor $(s',0)(b,e) \in I \propto F$ a contradiction. By similarly way, we get $b \in Ann(F)$.

Conversely, let $(0,0) \neq (a,e)(b,f) \in I \propto F$ so $ab \in I$, two case are then possible.

Case 1: If $ab \neq 0$ there exists $s' \in S$ such that $s'a \in I$ or $s'b \in I$. Assume that $s'a \in I$. If $s'e \in F$, then $(s',0)(a,e) = (s'a, s'e) \in I \propto F$ as desired. Assume that $s'e \notin F$, if $sE \subset F$ for some $s \in S$, $s'', s', e \in F$, put $s = s''s'$, so $sa = s''s'a \in I$ since $s'a \in I$ and $se = s's' \in F$, then we get $(s,0)(a,e) \in I \propto F$. Assume that $sE \nsubseteq F$ for every $s \in S$ then $s'^2I^2 = 0$ for some $s' \in S$, we put that $b' = s'b$. If $s'b \in I$, then $s'^2ab \in s'^2I^2 = 0$ and so $ab' = 0$. Hence $s'ab = 0$. It follows that $ab = 0$ since s is a regular element, a contradiction with the fact that $s'b' \notin I$. So suppose that $s'e \notin F$. By (3), we have $b'e \notin F$ and so $s'be \notin F$, we get also $s'a \in I$, then $s'af \in F$, $(IE \subseteq F)$ and we have $s'af + s'be \in F$, a desired contradiction.

Case 2: If $ab = 0$. Suppose that $s'e$ and $s'f \notin F$. By (4) we have $af + be = 0$ or $af + be \notin F$ a contradiction. Then $s'e \in F$ or $s'f \in F$. On the other hand, assume that neither $s'a \in I$ nor $s'b \in I$. If $s'e \in F$ and $s'f \in F$, then by (4) $s'(af + be) = 0$ and so $af + be = 0$ since $S \cap Ann(E) = \emptyset$, again a contradiction. Hence without loss of generality, we may assume that $s'e \in F$ and $s'f \notin F$. By (4) and (2), we get $s'be = 0$ and $s'af = 0$ or $s'af \notin F$, a contradiction. Then

$s'a \in I$ or $s'b \in I$. Assume that $s'a \in I$ and $s'e \notin F$, then $s'f \in F$. But if $s'a \in I$, then $(s', 0)(b, f) \in I \propto F$. Now if $s'b \notin I$, then by (4) we have $af = 0$. Since $s'e \notin F$ by (2), either $s'be = 0$ or $s'be \notin F$, a contradiction. Hence $s'e \in F$. Therefore, $(s', 0)(e, f) \in I \propto F$, as desired. □

A submodule F of E satisfies $(*)$ if at least one of the three conditions (2-3-4) of Theorem 2.13 is not hold for every S-weakly prime ideal I of A, (where S is a multiplicative subset of A); (i.e., $I \propto F$ is not $S \propto E$-weakly prime). Also, we say that a trivial extension satisfies $(**)$, if every ideal of $A \propto E$ is homogeneous; that is, the ideals of $A \propto E$ has the form $I \propto F$, where I is an ideal of A and F is submodule of E. Set, $T = \{I$ is a nonzero S-weakly prime. For every $s \in S$, $sa \notin I$, $sb \notin I$ and $ab = 0$ implies $a \in Ann(E)$ and $b \in Ann(E)\}$. The following theorem studies the possible transfer of the S-WP-ring property between a ring A and a trivial ring extension $A \propto E$.

Theorem 2.14. *Let A be a ring, E be a nonzero A-module and F a submodule of B. Let $S \subset Reg(A)$ be a multiplicative subset of A such that $S \cap Ann(F) = \emptyset$. Then:*

(1) If $A \propto E$ is a $(A \propto E)$-WP-ring, then F satisfies both $(*)$ and $sE \not\subset F$ for every $s \in S$, and every ideal in T is S-prime.

(2) Assume that $A \propto E$ satisfies (**) and A is a S-WP-ring. Then $A \propto E$ is a $(A \propto E)$-WP-ring if and only if the following condition holds: $F \subseteq E$ satisfying (*).

To prove Theorem 2.14, we need the following two Lemmas.

Lemma 2.15. *Let A be a ring, I be an ideal of A and E be an A-module. Then $I \propto E$ is $(S \propto E)$-prime if and only if I is S-prime.*

Proof. Assume that $I \propto E$ is $(S \propto E)$-prime. Let $a, b \in A$ such that $ab \in I$. Hence $(a, 0)(b, 0) = (ab, 0) \in I \propto E$ and so, there exists $(s, e) \in S \propto E$ such that $(s, e)(a, 0) \in I \propto E$ or $(s, e)(b, 0) \in I \propto E$, then either $sa \in I$ or $sb \in I$ and hence I is S-prime.

Conversely, let $(a, e), (b, f) \in A \propto E$ such that $(a, e)(b, f)) = (ab, af + be) \in I \propto E$. Thus $ab \in I$ and so either $sa \in I$ or $sb \in I$ for some $s \in S$. Consequently, $(s, 0)(b, e) \in I \propto E$ or $(s, 0)(b, f) \in I \propto E$, as desired. □

Lemma 2.16. *Let A be a ring, I be an ideal of A, E be an A-module and F a submodule of E. Then $I \propto F$ is an $(S \propto E)$-prime if and only if I is S-prime and $sE \subset F$ for some $s \in S$.*

Proof. Assume that $I \propto F$ is an $(S \propto E)$-prime. Let $a, b \in A$ such that $ab \in I$. Hence $(a, 0)(b, 0) \in I \propto F$ and there exists $(s, e) \in S \propto E$

such that $(s,e)(a,0) \in I \propto F$ or $(s,e)(b,0) \in I \propto F$. Then either $sa \in I$ or $sb \in I$ and so I is S-prime. Now we claim that $sE \subset F$ for some $s \in S$. Let $e' \in E$, then $(0,e')(0,e') = (0,0) \in I \propto F$ and there exists $(s,e) \in S \propto E$ such that $(s,e)(0,e') \in I \propto F$. Consequently $(0,se) \in I \propto F$ and then $se \in F$. Hence $sE \subset F$ for some $s \in S$.

Conversely. Let $(a,e),(b,f) \in A \propto F$ such that $(a,e)(b,f) \in I \propto F$, in particular we get $ab \in I$. But There exists $s \in S$ such that either $sa \in I$ or $sb \in I$. So $se, sf \in F$ since $sE \subset F$ for some $s \in S$. Thus either $(s,0)(a,e) \in I \propto F$ or $(s,0)(b,f) \in I \propto F$, as desired. □

Proof of Theorem 2.14

(1) Assume that there exists a submodule F which satisfies the condition ($sE \not\subset F$ for every $s \in S$) but does not satisfy the $(*)$-condition. Then $I \propto F$ is a $(S \propto E)$-weakly prime ideal for some S-weakly prime I of A. By hypothesis, $I \propto F$ is $(S \propto F)$-prime, a desired contradiction by Lemma 2.16 above. Now, let I in T. By [18, Theorem 3.1], $I \propto E$ is $(S \propto E)$-weakly prime and hence $(S \propto E)$-prime. Then I is S-prime by Lemma 2.15 above, as desired.

(2) Assume that $A \propto E$ is a $(S \propto E)$-WP-ring. By (1), F which satisfies both the conditions ($sE \not\subset F$ for every $s \in S$) and $(*)$. Let H be a nonzero $(S \propto E)$-weakly prime ideal of $A \propto E$. Since F satisfies the two conditions above, we get $H = I \propto E$. But $I \propto E$ is $(S \propto E)$-weakly prime by Theorem 2.13 above, then I must to be S-weakly prime and so I is S-prime since A is assumed to be S-WP-ring. Following Lemma 2.15, we get $I \propto E$ is $(S \propto E)$-prime, as desired.

Example 2.17. *Let A be an integral domain with quotient field K and E be a K-vector space; such that $dim_K(E) > 1$. Then, $A \propto E$ is not an $(S \propto E)$-WP-ring.*

Proof. Let F be a K-vector subspace of E. By [10, Corollary 3.2], $0 \propto F$ is weakly prime, then it is $(S \propto E)$-weakly prime. Hence, E does not satisfy (*). Following Theorem 2.14, $A \propto E$ is not $(S \propto E)$-WP-ring. □

Proposition 2.18. *Let $(A, \mathfrak{m})$ be a local ring, E be an A-module such that $\mathfrak{m}E = 0$. If E is a simple A-module, then $A \propto E$ is a $(S \propto E)$-WP-ring.*

Proof. Assume that there is a nonzero $(S \propto E)$-weakly prime ideal H which is not $(S \propto E)$-prime. By [1, Proposition 2.4], $H \subsetneq \text{Nil}(A \propto E) = 0 \propto E$. Then $H = 0 \propto F$, where $F \subsetneq E$, a desired contradiction and this completes the proof. □

Using the above result, we can construct new and no trivial examples of S-WP-rings.

Example 2.19. *Let A be a local domain with maximal ideal $\mathfrak{m}$. Then, $A \propto (A/\mathfrak{m})$ is an $(S \propto (A/\mathfrak{m}))$-$WP$-ring.*

The following Proposition 2.11 study the S-weakly prime ideal in the trivial rings extension under some conditions.

Proposition 2.20. *Let D be an integral domain and Q is a divisible D-module and S be a multiplicative subset of D. Let N be a D-submodule of Q and I be an ideal of D. Then:*

(1) $I \propto Q$ is $(S \propto Q)$-weakly prime if and only if I is S-weakly prime.
(2) If there exists $s \in S$ such that $sQ \subset N$, then $0 \propto N$ is $(S \propto Q)$-weakly prime.
(3) If Q/N is S-torsion free D-module, then the following are equivalent:
   (a) $0 \propto N$ is an $(S \propto Q)$-weakly prime,
   (b) $0 \propto N$ is weakly prime.

Before establishing Proposition 2.20, we need the following Lemma 2.21

Lemma 2.21. *Let A be a ring, S be a multiplicative subset of A and M be an A-module. Then for every A-module X, X is an S-torsion free A-module if and only if X is an $(S \propto M)$-torsion free $(A \propto M)$-module.*

Proof. If X is an S-torsion free A-module. We claim that X is an $(S \propto M)$-torsion free $(A \propto M)$-module. Let $(s, e) \in S \propto M$ and $x \in X$ such that $(s, e)x = 0$, then $sx = 0$ and so $x = 0$, as desired.

Conversely, if X is an $(S \propto M)$-torsion free $(A \propto M)$-module, then for every $s \in S$ and $x \in X$ such that $sx = 0$, we get $(s, 0)x = 0$ and so $x = 0$, as desired X is an S-torsion free A-module. □

Proof of Proposition 2.20

(1) Follows immediately from [2, Corollary 3.3].

(2) This is obvious by definition of S-weakly prime.

(3) If Q/N is an S-torsion free D-module, then so is $D \propto (Q/N)$, i.e., $D \propto (Q/N) \cong \frac{D \propto Q}{0 \propto N}$ is an S-torsion free D-module. Following Lemma 2.21, we get $D \propto (Q/N) \cong \frac{D \propto Q}{0 \propto N}$ is an $(S \propto Q)$-torsion free $(D \propto Q)$-module. The equivalence $(a) \iff (b)$ follows immediately from Theorem 2.12.

Next, in the last part of this paper, we study the transfer of S-WP-rings in amalgamation of rings along an ideal. The following remarks investigate the trivial case.

Remark 2.22. *Let $f : A \longrightarrow B$ be a rings homomorphism and J be an ideal of B. If $J = B$, then $A \bowtie^f J$ is always an S-WP-ring by Remark 2.2.*

Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. Set $V = \{I$ is a nonzero S-weakly prime ideal of $A \mid ab = 0$ and $sa \notin I,\ sb \notin I$ for each $s \in S$then$f(a)j + f(s)f(b)i + ij = 0$ for every $i, j \in J\}$ and $S' = \{(s, f(s) \mid s \in S\}$. Clearly $I \bowtie^f J$ is an ideal of $A \bowtie^f J$ and S' is a multiplicative set of $A \bowtie^f J$.

Theorem 2.23. *With the above notation, if $A \bowtie^f J$ is a S-WP-ring, then every ideal of V is S-prime.*

For establishing Theorem 2.23, we need the following Lemma 2.24.

Lemma 2.24. *Let $f : A \longrightarrow B$ be a rings homomorphism and S be a multiplicative subset of A and J be an ideal of B. Then $I \bowtie^f J$ is S'-prime if and only if I is S-prime.*

Proof. Assume that $I \bowtie^f J$ is S'-prime. Let $a, b \in A$ with $ab \in I$, then $(a, f(a))(b, f(b)) \in I \bowtie^f J$. So there exists $s \in S$ such that $(s, f(s))(a, f(a)) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)) \in I \bowtie^f J$. Then $sa \in I$ or $sb \in I$.

Conversely, let $(a, f(a) + i), (b, f(b) + j) \in A \bowtie^f J$ with $(a, f(a) + i)(b, f(b) + j) \in I \bowtie^f J$, then $ab \in I$. So there exists $s \in S$ such that $sa \in I$ or $sb \in I$. We easily get that $(s, f(s))(a, f(a) + i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b) + j) \in I \bowtie^f J$. We are done. □

Proof of Theorem 2.23

Let I be an ideal of V. Then $I \bowtie^f J$ is an S-weakly prime ideal of $A \bowtie^f J$. In fact, let $(0, 0) \neq (a, f(a) + i)(b, f(b) + j) \in I \bowtie^f J$, then $ab \in I$. If $ab \neq 0$, we get either $sa \in I$ or $sb \in I$ for some $s \in S$. Hence $(s, f(s))(a, f(a) + i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b) + j) \in I \bowtie^f J$. Now assume that $ab = 0$ where $sa \neq I$ and $sb \neq I$ for each $s \in S$. Then $f(a)j + f(b)i + ij = 0$ for each $i, j \in J$. Since I is an ideal of V, a contradiction with the fact that $(ab, f(a)j + f(b)i + ij) \neq (0, 0)$. But $A \bowtie^f J$ is an S-WP-ring we get $I \bowtie^f J$ is S-prime, then I is S-prime by Lemma 2.24 above.

Theorem 2.25. *Let $f : A \to B$ be a ring homomorphism where A is an integral domain and let J be a regular ideal of B such that $f^{-1}(J) \neq 0$. If $S' \subset Reg(A \bowtie^f J)$, then $A \bowtie J$ is an S'-WP-ring.*

Proof. Assume that there is a nonzero weakly prime ideal H of $A \bowtie^f J$ that is not S'-prime. By [1, Proposition 2.4], $H \subsetneq \mathrm{Nilp}\left(A \bowtie^f J\right) \subseteq \mathrm{Nilp}(A) \bowtie^f J$. Then, $H = 0 \times K$, where $K \subsetneq J$. Pick a nonzero element $a \in f^{-1}(J)$ and let j be regular element of J. As, $H^2 = 0$, we get $j \in J \backslash K$. Consider $0 \neq k \in K$, we have $(0,0) \neq (a,k)(0,j) \in 0 \times K$. But, neither $(s, f(s))(a,k) \in 0 \times K$ nor $(s, f(s))(0,j) \in 0 \times K$ for every $s \in S$, a contradiction. $\square$

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